Reference: Lawrence Perko
Differential equations and Dynamical systems.

2.10 Saddles, Nodes, Foci and Centers

(1a)

A linear system

$$\dot{x} = Ax \qquad (1)$$

where $x \in R^2$ is said to have a saddle, node, focus or center at the origin if its phase portrait is linearly equivalent to one of the phase portraits in figures 1-4 in Section 1.5 of chapter 1 respectively; i.e., if there exists a non-singular linear transformation which reduces the matrix A to one of canonical matrices B in Case I-IV of section 1.5 in chapter 1 respectively. (L. Perko).

Consider a nonlinear system

$$\dot{x} = f(x) \qquad (2)$$

is said to have a saddle, a sink or a source at a hyperbolic equilibrium point x_0 if the linear part of f at x_0 has eigenvalues with both positive and negative real parts, only has eigenvalues with negative real parts, or only has eigenvalues with positive real parts, respectively.

Here, we define the concept of a topological saddle for the nonlinear system (2) with $x \in R^2$ and show that if x_0 is a hyperbolic equilibrium point of (2) then it is a topological saddle if and only if it is a saddle of (2); i.e. a hyperbolic equilibrium point x_0 is a topological saddle for (2) if and only if the origin is a saddle for (1) with $A = Df(x_0)$.

In connection, we discuss topological saddles for nonhyperbolic equilibrium points of (2) with $x \in R^2$.

We also refine the classification of sinks of the non-linear system (2) into stable nodes and foci and show that, under slightly stronger hypotheses on the function f, i.e. stronger than $f \in C^1(E)$, a hyperbolic critical point x_0 is a stable node or focus for the nonlinear system (2) if and only if it is respectively a stable node or focus for the linear system (1) with $A = Df(x_0)$.

Similarly, a source of (2) is either an unstable node or focus of (2) as defined below. Finally, we define centers and center-foci for the non-linear system (2) and show that, under the addition of nonlinear terms, a center of the linear system (1) may become either a center, a center-focus, or a stable or unstable focus of (2).

Here, We define various types of equilibrium points for planar system (2), So, it is convenient to introduce polar coordinates (r,θ) and to rewrite the system (2) in polar coordinates.

Let us take $X = (x,y)^T$, $f_1(X) = P(x,y)$
and $f_2(X) = Q(x,y)$.

Then, the non-linear system (2) can be written as

$$\dot{x} = P(x,y)$$
$$\dot{y} = Q(x,y) \quad \text{——— (3)}$$

if we take $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, then we have

$$r\dot{r} = x\dot{x} + y\dot{y}$$

and $$r^2\dot{\theta} = x\dot{y} - y\dot{x} \quad (3)$$

since $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\dot{\theta} = \frac{1}{1+\frac{y^2}{x^2}} \frac{d\left(\frac{y}{x}\right)}{} = \frac{x^2}{x^2+y^2} \frac{d\left(\frac{y}{x}\right)}{}$

$\dot{\theta} = \frac{x^2}{x^2+y^2}\left(\frac{x\dot{y}-y\dot{x}}{x^2}\right)$

$\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{r^2}$

It follows that $r>0$, the nonlinear system (3) can be written in terms of polar coordinates as

$$\dot{r} = P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta \quad (4)$$
$$r\dot{\theta} = Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta$$

Or as

$$\frac{dr}{d\theta} = F(r,\theta) = \frac{r\left[P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta\right]}{Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta} \quad (5)$$

Writing the system of differential equations (3) in polar coordinates will often reveal the nature of the equilibrium point or critical point at the origin.

Example 1 Consider the system

$$\dot{x} = -y - xy$$
$$\dot{y} = x + x^2$$

Here, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Solution. in polar coordinates. For $r>0$, we have

since.
$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$$

$$\dot{r} = \frac{x(-y-xy) + y(x+x^2)}{r}$$

or
$$\dot{r} = \frac{-\cancel{xy} - \cancel{x^2y} + \cancel{xy} + \cancel{x^2y}}{r}$$

or $\dot{r} = 0$

Thus, $\boxed{r = \text{constant}}$

and
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x(x+x^2) - y(-y-xy)}{r^2} \qquad \because r^2 = x^2+y^2$$

or
$$\dot{\theta} = \frac{x^2 + x^3 + y^2 + xy^2}{x^2+y^2} = \frac{(x^2+y^2) + x(x^2+y^2)}{x^2+y^2}$$

or
$$\dot{\theta} = \frac{(x^2+y^2)(1+x)}{(x^2+y^2)} = 1+x > 0$$

or $\boxed{\dot{\theta} = 1+x > 0}$ for $x > -1$. Thus, along any trajectory of this system in the half plane $x>-1$, $r(t)$ is constant and $\theta(t)$ increases without bound as $t \to \infty$. It means that, the phase portrait in a nbd of the origin is equivalent to the phase portrait in figure 1. Here, the origin is called a centre for the given nonlinear system

A center at the origin

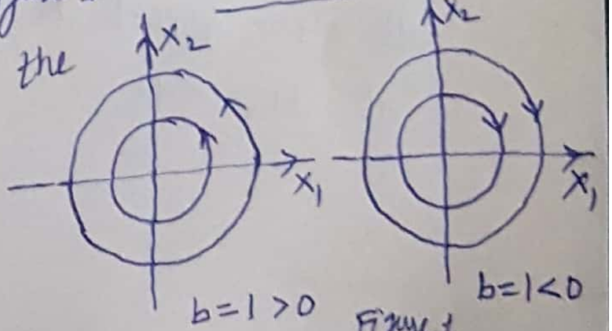


Figure 1.

Example 2

Consider the system

$$\dot{x} = -y - x^3 - xy^2$$
$$\dot{y} = x - y^3 - x^2 y$$

Solution

Here $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

In polar coordinates, for $r > 0$, we have

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{x(-y - x^3 - xy^2) + y(x - y^3 - x^2 y)}{r}$$

$$= \frac{\cancel{-xy} - x^4 - x^2y^2 + \cancel{xy} - y^4 - x^2y^2}{r}$$

$$= \frac{-x^4 - x^2y^2 - x^2y^2 - y^4}{r}$$

$$= \frac{-x^2(x^2+y^2) - y^2(x^2+y^2)}{r}$$

$$\dot{r} = \frac{-(x^2+y^2)(x^2+y^2)}{r}$$

or $\dot{r} = \frac{-r^2 \cdot r^2}{r} = \frac{-r^4}{r}$ $\quad \{\because r^2 = x^2 + y^2$

or $\boxed{\dot{r} = -r^3}$

so that $\frac{dr}{r^3} = -dt$

on integration $\frac{r^{-2}}{-2} = -t + C$

$r^{-2} = 2t - 2C$

$\frac{1}{r^2} = 2t - 2C$ or $r^2 = \frac{1}{(2t - 2C)}$

At $t=0$, $r=r_0$

$$r_0^2 = \frac{1}{-2c} \qquad \left\{\because r^2 = \frac{1}{2t-2c}\right.$$

$$c = -\frac{1}{2r_0^2}$$

Thus, $$r^2 = \frac{1}{2t+\frac{1}{r_0^2}} = \frac{r_0^2}{1+2r_0^2 t}$$

$$\boxed{r = r_0(1+2r_0^2 t)^{-1/2}} \quad \text{for } t > -\frac{1}{2r_0^2}$$

$\left\{\begin{array}{l} \text{since} \\ 1+2r_0^2 t > 0 \\ 2r_0^2 t > -1 \\ \text{for } \quad t > -\frac{1}{2r_0^2} \end{array}\right.$

and $$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x(x-y^3-x^2y) - y(-y-x^3-xy^2)}{r^2}$$

$$= \frac{x^2 - xy^3 - x^3y + y^2 + x^3y + xy^3}{r^2}$$

$$= \frac{x^2+y^2 - xy^3 + xy^3}{r^2} = \frac{x^2+y^2}{r^2}$$

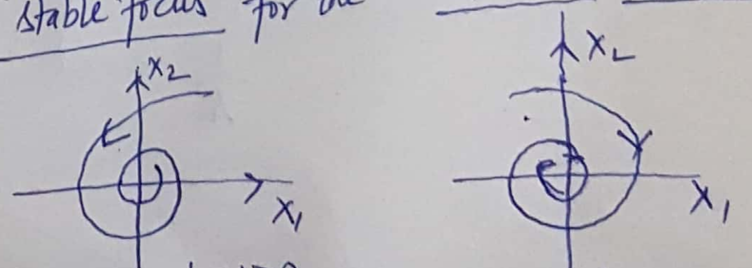
$$\dot{\theta} = \frac{r^2}{r^2} = 1 \qquad (\because x^2+y^2 = r^2)$$

$$\boxed{\dot{\theta} = 1} \qquad \boxed{\theta(t) = \theta_0 + t}$$

Here, we see that $r(t) \to 0$ and $\theta(t) \to \infty$ as $t \to \infty$ and the phase portrait for this system a nbd of the origin is qualitatively equivalent to the first/second figure 2.

Hence, the origin is called a 'stable focus' for the given non-linear system.

A stable focus at the origin



$b=1>0$ Figure 2 $b=1<0$